

# New asymptotic formulas for sums over zeros of functions from the Selberg class

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## Abstract

In this paper, new asymptotic formulas for sums over zeros of functions from the Selberg class are obtained. These results continue the investigations of Murty & Perelli [12], of Murty & Zaharescu [13], of Kamiya & Suzuki [8], of Steuding [19] and other authors.

Résumé. Dans ce article, nouvelles formules asymptotiques pour des sommes sur les zéros de fonctions de la classe de Selberg sont obtenus. Ces résultats continuent les recherches de Murty & Perelli [12], de Murty & Zaharescu [13], de Kamiya & Suzuki [8], de Steuding [19] et d'autres auteurs.

## 1 Introduction

Let  $\rho = \beta + i\gamma$  be a non-trivial zero of some function in the Selberg class. The aim of this paper is to give an asymptotic formula for sums involving zeros of functions in the Selberg class  $S$ , that is,

$$\sum_{\rho} e^{u\rho^2 - v\rho}, \text{ where } u > 0 \text{ and } v \in \mathbb{R}.$$

Furthermore, we will discuss a more general quantity

$$\sum_{\rho} e^{v(\rho - \frac{1}{2})} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho - \frac{1}{2})} dx,$$

for fixed  $v$  and  $u \rightarrow 0^+$ , where  $f$  belongs to a certain class and show asymptotic results for it. For this purpose, we use the Weil explicit formulas.

We refer to the survey of Kaczorowski and Perelli [6] and Selberg [18], for the definition and notations of the Selberg class. The Selberg class  $S$  consists of function  $F(s)$  of a complex variable  $s$  satisfying the following properties:  
*i)* (Dirichlet series) For  $\Re(s) > 1$ ,

$$F(s) = \sum_{n=1}^{+\infty} a_F(n) n^{-s},$$

where  $a(1) = 1$ .

*ii)* (Analytic continuation) For some integer  $m \geq 0$ ,  $(s-1)^m F(s)$  extends to an entire function of finite order. We denote by  $m_F$  the smallest integer  $m$  which satisfies this condition .

*iii)* (Functional equation) There are numbers  $Q > 0$ ,  $\lambda_j > 0$  and  $\mu_j \in \mathbb{C}$  with  $\Re(\mu_j) \geq 0$ , so that

$$\phi(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) F(s)$$

satisfies

$$\phi(s) = \omega \bar{\phi}(1-s),$$

where  $\omega$  is a complex number with  $|\omega| = 1$  and  $\bar{\phi}(s) = \overline{\phi(\bar{s})}$ .

*iv)* (Euler product)

$$F(s) = \prod_p F_p(s),$$

where

$$F_p(s) = \exp \left( \sum_{n=1}^{+\infty} \frac{b(p^k)}{p^{ks}} \right),$$

and  $b(p^k) = O(p^{k\theta})$  for some  $\theta < \frac{1}{2}$  and  $p$  denotes a prime number.

*v)* ( Ramanujan Hypothesis ) For any fixed  $\epsilon > 0$ ,  $a(n) = O(n^\epsilon)$ .

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, i.e, that all non trivial (non-real) zeros lie on the

critical line  $\Re(s) = \frac{1}{2}$ . The degree of  $F(s) \in S$  is defined by

$$d_F = 2 \sum_{j=1}^r \lambda_j.$$

The logarithmic derivative of  $F(s)$  also has the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s},$$

where  $\Lambda_F(n) = b(n) \log n$  is an analogue of the Von Mangoldt function  $\Lambda(n)$  defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$q_F = \frac{(2\pi)^{d_F} Q^2}{\beta}, \quad \text{where } \beta = \prod_{j=1}^r \lambda_j^{-2\lambda_j},$$

be the conductor (or modulus) of  $F \in S$ .

The first main result of this paper is stated in the following Theorem.

**Theorem 1.1.** *i) Let  $T > 1$ . For  $v = u$  or  $v = 0$ , we have:*

$$\sum_{|\gamma| \leq T} e^{u\rho^2 - v\rho} = \frac{d_F}{\sqrt{16\pi u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \frac{1}{\sqrt{4\pi u}} \log \left( \frac{q_F}{(4\pi)^{d_F}} \right) + O \left( \frac{e^{-uT^2}}{T} (\log T)^2 \right), \quad \text{if } u \rightarrow 0^+,$$

where  $\gamma_0$  is the Euler constant,  $\rho$  runs over all non-trivial zeros of  $F(s)$  counted with multiplicity and the  $O$ -term depends on  $\mu_j$ .

ii) Let  $T > 1$ . For any integer  $m \geq 2$ , we have

$$\sum_{|\gamma| \leq T} e^{u\rho^2 + (\log m)\rho} = -\frac{\Lambda_F(m)}{\sqrt{4\pi u}} + O \left( \frac{e^{-uT^2}}{T} (\log T)^2 \right), \quad \text{if } u \rightarrow 0^+,$$

and

$$\sum_{|\gamma| \leq T} e^{u\rho^2 - (\log m)\rho} = -\frac{\overline{\Lambda_F(m)}}{m\sqrt{4\pi u}} + O \left( \frac{e^{-uT^2}}{T} (\log T)^2 \right), \quad \text{if } u \rightarrow 0^+,$$

where the  $O$ -term depends on  $\mu_j$  and  $m$ .

In this direction, assuming the Generalized Riemann Hypothesis (GRH), Murty and Perelli [12] proved that, if  $F \in S$ , then for  $T > 1$  and  $n \in \mathbb{N}^*$

$$\sum_{|\gamma| \leq T} n^\rho = -\frac{T}{\pi} \Lambda_F(n) + O(n^{3/2} \log T), \quad (1)$$

which is an extension to the Selberg class  $S$  of the uniform version of Landau's formula obtained by Gonek [3]. Murty and Zaharescu [13] proved unconditionally that, if

$F \in S$ ,  $x \geq 2$ ,  $\epsilon > 0$ ,  $n \in \mathbb{N}$  and  $n \geq x$ , then

$$\begin{aligned} \sum_{|\gamma| \leq T} x^\rho &= -\frac{\Lambda_F(n)}{\pi} \frac{\sin(T \log \frac{T}{n})}{\log \frac{x}{n}} \\ &+ O_{\epsilon, F}(x^{1+\epsilon} \log^2 T) + O\left(n^{1+\theta} \sum_{|n-p^k| < n^\theta; p < p(\epsilon, F)} \frac{1}{|n-p^k|}\right), \end{aligned}$$

where  $\rho$  runs over all non-trivial zeros of  $F(s)$  and  $p(\epsilon, F)$  depend uniquely on  $F$  and  $\epsilon$ .

**Corollary 1.2.** *For  $v = u$  or  $v = 0$ , we have*

$$\sum_{\rho} e^{u\rho^2 - v\rho} = \frac{d_F}{\sqrt{16\pi u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \frac{1}{\sqrt{4\pi u}} \log \left( \frac{q_F}{(4\pi)^{d_F}} \right) + O(1), \text{ if } u \rightarrow 0^+,$$

where the sum  $\sum_{\rho}$  runs over all non-trivial zeros of  $F(s)$  counted with multiplicity and the  $O$ -term depends on  $\mu_j$ .

In the case of the classical Riemann zeta function, we have

$$\sum_{\rho} e^{u\rho^2 - v\rho} = \frac{1}{\sqrt{16\pi u}} \log \frac{1}{u} - \frac{\log(16\pi^2) + \gamma_0}{\sqrt{16\pi u}} + O(1), \text{ if } u \rightarrow 0^+,$$

which was established by Kamiya and Suzuki in [8].

The main tool which will be used in the proofs of the second main result (Theorem 1.4 below) is the Weil explicit formulas given by the following Proposition.

**Proposition 1.3. (Omar-Mazhouda [14] [15])** *Let  $f$  be some complex-valued function on  $\mathbb{R}$  satisfying the conditions :*  
*a)  $f$  is normalized, that is,*

$$f(x) = \frac{f(x^+) + f(x^-)}{2}, \quad x \in \mathbb{R},$$

*where  $f(x^+)$  ( resp.  $f(x^-)$ ) means the right (resp. left) limit of  $f$ .*  
*b) There exist a constant  $b > 0$  such that*

$$V_{\mathbb{R}} \left( f(x) e^{(\frac{1}{2}+b)|x|} \right) < \infty,$$

*where  $V_{\mathbb{R}}(.)$  means the total variation on  $\mathbb{R}$ .*  
*c) There is a constant  $\epsilon > 0$  such that*

$$f(x) = \begin{cases} f(x^+) + O(|x|^\epsilon), & x \mapsto 0^+ \\ f(x^-) + O(|x|^\epsilon), & x \mapsto 0^-. \end{cases}$$

*Then*

$$\begin{aligned} \sum_{\rho} \int_{-\infty}^{+\infty} f(x) e^{x(\rho - \frac{1}{2})} dx \\ = & - \sum_{n=2}^{+\infty} \frac{\Lambda_F(n)}{\sqrt{n}} f(\log n) - \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{\sqrt{n}} f(-\log n) \\ & + m_F \left( \int_{-\infty}^{+\infty} f(x) e^{\frac{x}{2}} dx + \int_{-\infty}^{+\infty} f(x) e^{-\frac{x}{2}} dx \right) + 2f(0) \log Q \\ & + \sum_{j=1}^r \lambda_j \left( \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \overline{\mu_j} \right) \right) f(0) \\ & - \sum_{j=1}^r \lambda_j \int_0^{+\infty} (f(-\lambda_j x) - f(0^-)) \frac{e^{-(\frac{\lambda_j}{2} + \mu_j)x}}{1 - e^{-x}} dx \\ & - \sum_{j=1}^r \lambda_j \int_0^{+\infty} (f(\lambda_j x) - f(0^+)) \frac{e^{-(\frac{\lambda_j}{2} + \overline{\mu_j})x}}{1 - e^{-x}} dx. \end{aligned} \tag{2}$$

This is the so called Weil explicit formulas with Mestre's formulation [11]. Proposition 1.3 is proved by a way similar to the proof of the Weil explicit

formulas in [11]. There is no essential difference or difficulty in our case because of conditions for  $F(s)$ . Hence we omit the proof of Proposition 1.3.

Replacing  $f(x)$  in (2) by  $f(\frac{x-v}{u})$ , with the assumption

$$f\left(\frac{x-v}{u}\right) = \begin{cases} f((\frac{-v}{u})^+) + O(|x|^\epsilon), & x \mapsto 0^+ \\ f((\frac{-v}{u})^-) + O(|x|^\epsilon), & x \mapsto 0^-. \end{cases}$$

We can easily verify that  $f(\frac{x-v}{u})$  is normalized and that

$$V_{\mathbb{R}}\left(f\left(\frac{x-v}{u}\right)e^{(\frac{1}{2}+b)|x|}\right) < \infty.$$

Moreover, one obtains

$$\begin{aligned} \sum_{\rho} e^{v(\rho-\frac{1}{2})} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho-\frac{1}{2})} dx \\ = - \sum_{n=2}^{+\infty} \frac{\Lambda_F(n)}{\sqrt{n}} f\left(\frac{\log n - v}{u}\right) - \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{\sqrt{n}} f\left(\frac{-\log n - v}{u}\right) \\ + m_F \left( \int_{-\infty}^{+\infty} f\left(\frac{x-v}{u}\right) e^{\frac{x}{2}} dx + \int_{-\infty}^{+\infty} f\left(\frac{x-v}{u}\right) e^{-\frac{x}{2}} dx \right) \\ + 2f\left(\frac{-v}{u}\right) \log Q + \sum_{j=1}^r \lambda_j \left( \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \overline{\mu_j} \right) \right) f\left(\frac{-v}{u}\right) \\ - \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( f\left(\frac{-\lambda_j x - v}{u}\right) - f\left(\left(\frac{-v}{u}\right)^-\right) \right) \frac{e^{-(\frac{\lambda_j}{2} + \mu_j)x}}{1 - e^{-x}} dx \\ - \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( f\left(\frac{\lambda_j x - v}{u}\right) - f\left(\left(\frac{-v}{u}\right)^+\right) \right) \frac{e^{-(\frac{\lambda_j}{2} + \overline{\mu_j})x}}{1 - e^{-x}} dx. \end{aligned} \quad (3)$$

Formula (3) is valid for  $v = 0$  and all  $0 < u < 1$  under conditions a), b) and c). Furthermore it is valid for all  $0 < u < 1$  and  $v \in \mathbb{R}$  under conditions a), b) and c') given by: there exist  $D > 0$  and  $\epsilon > 0$  such that for all  $a \in \mathbb{R}$

$$\begin{cases} |f(a+x) - f(a^+)| \leq D|x|^\epsilon, & x > 0 \\ |f(a+x) - f(a^-)| \leq D|x|^{\epsilon'}, & x < 0. \end{cases}$$

The second result of this paper gives an asymptotic to the left-hand side of (3) for fixed  $v$  and  $u \mapsto 0^+$ . It is stated in the following Theorem.

**Theorem 1.4.** *Let  $f$  be some complex-valued function on  $\mathbb{R}$  satisfying the above conditions a), b) and c). Then, when  $u \mapsto 0^+$ , one has*

$$\sum_{\rho} e^{v(\rho-\frac{1}{2})} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho-\frac{1}{2})} dx = \begin{cases} O(u) & \text{if } v \neq \pm \log m, m = 1, 2, \dots \\ -\frac{\Lambda_F(m)}{\sqrt{m}} f(0) + O(u) & \text{if } v = \log m, m = 2, 3, \dots \\ -\frac{\Lambda_F(m)}{\sqrt{m}} f(0) + O(u) & \text{if } v = -\log m, m = 2, 3, \dots \end{cases}$$

When  $v = 0$ , one gets

$$\begin{aligned} \sum_{\rho} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho-\frac{1}{2})} dx &= -f(0) \left[ 2 \log Q + \sum_{j=1}^r \lambda_j \log \left| \left( \frac{\lambda_j}{2} + \mu_j \right) u^2 \right| \right] \\ &\quad - 2 \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( \frac{f(\lambda_j x) + f(-\lambda_j x)}{2} - f(0) e^{-x} \right) \frac{dx}{x} + O(u), \end{aligned}$$

where  $\Lambda_F(m) = b(n) \log n$  is the generalized von Mangoldt function.

Theorem 1.4 says that the quantity

$$\sum_{\rho} e^{v(\rho-\frac{1}{2})} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho-\frac{1}{2})} dx$$

behaves quite differently whether  $v$  is in  $\{0\} \cup \{\log p^l; p : \text{prime}, l : \text{positive integer}\}$  or not.

Now, we give an expression for the error term in Theorem 1.4 under the Riemann Hypothesis.

**Theorem 1.5.** *Let  $f$  be some complex-valued function on  $\mathbb{R}$  satisfying the conditions :*

- a)  $f$  is normalized.
- b) There exists a constant  $b > 0$  such that

$$V_{\mathbb{R}} \left( f(x) e^{(\frac{1}{2}+b)|x|} \right) < \infty,$$

where  $V_{\mathbb{R}}(.)$  means the total variation on  $\mathbb{R}$ .

- c) There exists a constant  $\epsilon > 0$  such that

$$f(x) = \begin{cases} f(x^+) + O(|x|^\epsilon), & x \mapsto 0^+, \\ f(x^-) + O(|x|^\epsilon), & x \mapsto 0^-. \end{cases}$$

d) For the Fourier transform  $\widehat{f}$  of  $f$  defined by

$$\widehat{f}(t) = \int_{-\infty}^{+\infty} f(x)e^{-itx} dx,$$

it holds that

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)| |\log |t|| dt < \infty.$$

Let  $u$  be such that  $0 < u < 1$ . Then the error term in theorem 1.1, for the case  $v = 0$ , can be expressed in the form

$$-u \int_0^{+\infty} \left( N_F(T) - \frac{d_F}{2\pi} T \log \left( \frac{T}{2\pi} \right) - \frac{1}{2\pi} T (\log q_F - d_F) \right) \frac{d}{dT} \left( \widehat{f}(-uT) + \widehat{f}(uT) \right) dT,$$

where  $N_F(T)$  is the number of non-trivial zeros counted with multiplicities in the rectangle  $0 \leq \sigma \leq 1$  and  $0 \leq t \leq T$ .

We finish this section by giving the following Lemma.

**Lemma 1.6.** *Let  $f$  be some complex-valued function on  $\mathbb{R}$  satisfying conditions a), b), c) and d) of Theorem 1.5. Then*

$$\frac{1}{\pi\lambda} \int_0^{+\infty} \widehat{f}\left(-\frac{t}{\lambda}\right) \log t dt = - \int_0^{+\infty} \left( \frac{f(\lambda x) + f(-\lambda x)}{2} - f(0)e^{-x} \right) \frac{dx}{x}.$$

*Proof.* We evaluate the following double integral

$$I = \frac{1}{\pi} \int_0^{+\infty} \left( \int_0^{+\infty} f(\lambda y) \frac{x}{x^2 + y^2} dy - \frac{\pi}{2} f(\lambda x) \right) \frac{dx}{x}$$

by two ways. The first way is to use the equation

$$\frac{2}{\pi} \int_0^{+\infty} \frac{x}{x^2 + y^2} dy = 1.$$

From this and after some calculations, it follow that  $I = 0$ . The second way is to use the equation

$$\frac{1}{2} \int_{-\infty}^{+\infty} e^{-x|t| - ity} dt = \frac{x}{x^2 + y^2}, \quad x > 0.$$

From this it follow that

$$I = -\frac{1}{\pi\lambda} \int_0^{+\infty} \widehat{f}\left(-\frac{t}{\lambda}\right) \log t dt - \int_0^{+\infty} \left( \frac{f(\lambda x) + f(-\lambda x)}{2} - f(0)e^{-x} \right) \frac{dx}{x}.$$

□



## 2 Lemmas and proof of the Theorem 1.1

First, we give an explicit formula which connects certain sums involving  $\rho$  with sums involving prime numbers as in Proposition 1.3.

**Lemma 2.1.** *Let  $u > 0$ ,  $v \in \mathbb{R}$  and  $T > 1$ . We have*

$$\begin{aligned} \sum_{|\gamma| \leq T} e^{u\rho^2 - v\rho} &= -(m_F - 1)e^{u-v} + \frac{2 \log Q}{\sqrt{4\pi u}} e^{-\frac{v^2}{4u}} - 1 \\ &\quad - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \Lambda_F(n) e^{-\frac{(v+\log n)^2}{4u}} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{n} e^{-\frac{(v-2u-\log n)^2}{4u}} \\ &\quad + \frac{e^{\frac{u}{4} - \frac{v}{2}}}{\pi} \sum_{j=1}^r \lambda_j \int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda_j}{2} + \mu_j \right) + i\lambda_j t \right| e^{-ut^2 + it(u-v)} dt \\ &\quad - \sum_{j=1}^r \lambda_j I(\lambda_j, \mu_j) + O\left( \frac{e^{-uT^2}}{T} (\log T)^2 \right), \end{aligned} \quad (4)$$

where

$$\begin{aligned} I(\lambda, \mu) &= \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-\mu x} \frac{e^{-(v+\lambda x)^2/4u}}{\sqrt{4\pi u}} dx \\ &\quad + \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-(\lambda+\bar{\mu}x)} \frac{e^{-(v-\lambda x)^2/4u}}{\sqrt{4\pi u}} dx \end{aligned}$$

and  $\rho$  runs over all non-trivial zeros of  $F(s)$  counted with multiplicity.

*Proof.* Since

$$\frac{1}{\sqrt{4\pi u}} e^{-(v+\log x)^2/4u} = \frac{1}{2i\pi} \int_{\Re(s)=1+\delta} e^{us^2 - vs} x^{-s} ds,$$

we have

$$\frac{1}{2i\pi} \int_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{F'}{F}(s) e^{us^2 - vs} ds = - \sum_{n=2}^{+\infty} \Lambda_F(n) \frac{1}{\sqrt{4\pi u}} e^{-(v+\log n)^2/4u}. \quad (5)$$

Therefore

$$\begin{aligned} \frac{1}{2i\pi} \int_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{F'}{F}(s) e^{us^2 - vs} ds &= \frac{1}{2i\pi} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{F'}{F}(s) e^{us^2 - vs} ds \\ &\quad + \sum_{|\gamma| \leq T} e^{u\rho^2 - v\rho} + (m_F - 1)e^{u-v} + O\left( \frac{e^{-uT^2}}{T} (\log T)^2 \right). \end{aligned} \quad (6)$$

To prove (6), we consider the square connecting  $2-iT$ ,  $2+iT$ ,  $-1+iT$ ,  $-1-iT$ . The level  $T$  is chosen in such way that, for  $T \rightarrow +\infty$ ,  $\frac{F'}{F}(\sigma+iT) \ll \log^2 T$  uniformly for  $\sigma \in [-1, 2]$  [12, Equation (3.3)]. Integrating along this path the function  $\frac{F'(s)}{F(s)} e^{us^2-vs}$ , we get the sum of residues. The integrals along horizontal lines contributes

$$\left| \frac{1}{2i\pi} \int_{-\delta-i\infty}^{1+\delta+i\infty} \frac{F'}{F}(s) e^{us^2-vs} ds \right| \ll (\log T)^2 \int_{-\delta}^{1+\delta} e^{u(\sigma^2-T^2)-v\sigma} d\sigma \ll \frac{e^{-uT^2}}{T} (\log T)^2,$$

with the choice of  $\delta = \frac{1}{T}$ , where  $T \rightarrow +\infty$  and the integrals along vertical lines converge to the integrals appearing in formula (6). By the functional equation of  $F(s)$ , the first term of the right-hand side of (6) is

$$\begin{aligned} & \frac{1}{2i\pi} \int_{-\delta-i\infty}^{-\delta+i\infty} \left[ -\frac{\overline{F'}}{\overline{F}}(1-s) - 2\log Q \right] e^{us^2-vs} ds \\ & - \sum_{j=1}^r \lambda_j \frac{1}{2i\pi} \int_{-\delta-i\infty}^{-\delta+i\infty} \left( \frac{\Gamma'}{\Gamma}(\lambda_j s + \mu_j) + \frac{\Gamma'}{\Gamma}(\lambda_j(1-s) + \overline{\mu}_j) \right) e^{us^2-vs} ds \\ & = \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{n} \frac{1}{\sqrt{4\pi u}} e^{-(v-2u-\log n)^2/4u} - \frac{2\log Q}{\sqrt{4\pi u}} e^{-v^2/4u} + 1 \\ & - \sum_{j=1}^r \lambda_j \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\Gamma'}{\Gamma}\left(\frac{\lambda_j}{2} + \mu_j + i\lambda_j t\right) + \frac{\Gamma'}{\Gamma}\left(\frac{\lambda_j}{2} + \overline{\mu}_j - i\lambda_j t\right) \right) e^{u(\frac{1}{2}+it)^2-v(\frac{1}{2}+it)} dt. \end{aligned}$$

Hence and from (5) we have

$$\begin{aligned} & \sum_{|\gamma| \leq T} e^{u\rho^2-v\rho} \\ & = -(m_F - 1)e^{u-v} + \frac{2\log Q}{\sqrt{4\pi u}} e^{-\frac{v^2}{4u}} - 1 \\ & - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \Lambda_F(n) e^{-\frac{(v+\log n)^2}{4u}} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{n} e^{-\frac{(v-2u-\log n)^2}{4u}} \\ & + \sum_{j=1}^r \lambda_j \frac{e^{\frac{u}{4}-\frac{v}{2}}}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\Gamma'}{\Gamma}\left(\frac{\lambda_j}{2} + \mu_j + i\lambda_j t\right) + \frac{\Gamma'}{\Gamma}\left(\frac{\lambda_j}{2} + \overline{\mu}_j - i\lambda_j t\right) \right) e^{-ut^2+it(u-v)} dt \\ & + O\left(\frac{e^{-uT^2}}{T} (\log T)^2\right). \end{aligned} \tag{7}$$

Denote by  $L$  the expression

$$\sum_{j=1}^r \lambda_j \frac{e^{\frac{u}{4}-\frac{v}{2}}}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j + i\lambda_j t \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \bar{\mu}_j - i\lambda_j t \right) \right) e^{-ut^2+it(u-v)} dt.$$

Using the formula

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-zx} dx, \quad \operatorname{Re}(z) > 0,$$

we obtain

$$\begin{aligned} L &= \frac{e^{\frac{u}{4}-\frac{v}{2}}}{2\pi} \sum_{j=1}^r \lambda_j \int_{-\infty}^{+\infty} \left( \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j + i\lambda_j t \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \bar{\mu}_j - i\lambda_j t \right) \right) e^{-ut^2+it(u-v)} dt \\ &= \frac{e^{\frac{u}{4}-\frac{v}{2}}}{\pi} \sum_{j=1}^r \lambda_j \int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda_j}{2} + \mu_j \right) + i\lambda_j t \right| e^{-ut^2+it(u-v)} dt \\ &\quad - \frac{e^{\frac{u}{4}-\frac{v}{2}}}{2\pi} \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) \left[ e^{-(\frac{\lambda_j}{2} + \mu_j)x} \int_{-\infty}^{+\infty} e^{-ut^2+it(u-v-\lambda_j x)} dt \right] dx \\ &\quad - \frac{e^{\frac{u}{4}-\frac{v}{2}}}{2\pi} \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) \left[ e^{-(\frac{\lambda_j}{2} + \bar{\mu}_j)x} \int_{-\infty}^{+\infty} e^{-ut^2+it(u-v+\lambda_j x)} dt \right] dx. \end{aligned}$$

Note that

$$\int_{-\infty}^{+\infty} e^{-ut^2+it(u-v-\lambda_j x)} dt = \sqrt{\frac{\pi}{u}} e^{-(u-v-\lambda_j x)^2/4u} = \sqrt{\frac{\pi}{u}} e^{-\frac{u}{4}+\frac{v}{2}+\frac{\lambda_j}{2}x} e^{-(v+\lambda_j x)^2/4u}$$

and

$$\int_{-\infty}^{+\infty} e^{-ut^2+it(u-v+\lambda_j x)} dt = \sqrt{\frac{\pi}{u}} e^{-(u-v+\lambda_j x)^2/4u} = \sqrt{\frac{\pi}{u}} e^{-\frac{u}{4}+\frac{v}{2}-\frac{\lambda_j}{2}x} e^{-(v-\lambda_j x)^2/4u}.$$

Finally, by considering (7) and the above equations, Lemma 2.1 is proved.  $\square$

From the next Lemma given below, we will deduce Theorem 1.1.

**Lemma 2.2.** *For  $0 < u < 1$ , we have*

$$\int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2+it(u-v)} dt$$

$$= \begin{cases} O\left(\frac{1}{|u-v|^2}\right) & \text{if } v \neq 0 \text{ and } v \neq u, \\ \sqrt{\frac{\pi}{4u}} \left(\log \frac{1}{u} - \gamma_0\right) + \sqrt{\frac{\pi}{u}} \log\left(\frac{\lambda^2}{4}\right) + O(1) & \text{if } u = v \text{ or } v = 0, \end{cases}$$

where  $O(1)$  is a constant depending on  $\mu$ .

*Proof.* . Without loss of generality, we assume that  $\mu$  is a real number.

• In the case  $u = v$  and  $v \neq 0$ , we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2 + it(u-v)} dt \\ &= \int_0^{+\infty} \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) \frac{dt}{\sqrt{u}} \\ &= -\frac{1}{u-v} \int_0^{+\infty} \frac{d}{dt} \left[ \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} \right] \sin \left( \frac{t(u-v)}{\sqrt{u}} \right) dt \\ &= -\frac{\sqrt{u}}{(u-v)^2} \int_0^{+\infty} \frac{d^2}{dt^2} \left[ \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} \right] \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt \\ &= -\frac{\sqrt{u}}{(u-v)^2} \{M_1 + M_2 + M_3\}, \end{aligned} \tag{8}$$

where

$$M_1 = \int_0^{+\infty} \frac{2\lambda \left( -\lambda^2 t^2 + u \left( \frac{\lambda}{2} + \mu \right)^2 \right)}{\left( \lambda^2 t^2 + u \left( \frac{\lambda}{2} + \mu \right)^2 \right)^2} e^{-t^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt,$$

$$M_2 = -\int_0^{+\infty} \frac{8\lambda t^2}{\lambda^2 t^2 + u \left( \frac{\lambda}{2} + \mu \right)^2} e^{-t^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt$$

and

$$M_3 = \int_0^{+\infty} \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} (4t^2 - 2) \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt.$$

We have

$$\begin{aligned} |M_1| &\leq 2 \int_0^{+\infty} \frac{e^{-t^2}}{\lambda^2 t^2 + u \left( \frac{\lambda}{2} + \mu \right)^2} dt = \frac{2}{\lambda^2} \int_0^{+\infty} \frac{e^{-t^2}}{t^2 + u \left( \frac{1}{2} + \frac{\mu}{\lambda} \right)^2} dt \\ &\leq \frac{2}{\lambda^2} \frac{1}{u \left( \frac{1}{2} + \frac{\mu}{\lambda} \right)^2} \int_0^{\sqrt{u}} dt + 2 \int_{\sqrt{u}}^{+\infty} \frac{dt}{t^2} \\ &\ll \frac{1}{\sqrt{u}}, \end{aligned} \tag{9}$$

$$|M_2| \leq 8 \int_0^{+\infty} e^{-t^2} dt = 4\sqrt{\pi} \quad (10)$$

and

$$\begin{aligned} |M_3| &\leq \int_0^{+\infty} \left| \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) \right| e^{-t^2} (4t^2 + 2) dt \\ &\leq \log \frac{1}{u} \int_0^{+\infty} e^{-t^2} (4t^2 + 2) dt \\ &\quad + \int_0^{+\infty} \left| \log \left( u \left( \frac{\lambda}{2} + \mu \right)^2 + \lambda^2 t^2 \right) \right| e^{-t^2} (4t^2 + 2) dt \\ &\ll \log \frac{1}{u} + \int_{1/\lambda_j}^{+\infty} \log t e^{-t^2} (4t^2 + 2) dt \\ &\ll \log \frac{1}{u}. \end{aligned} \quad (11)$$

From (8), (9), (10) and (11), we obtain

$$\int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2 + it(u-v)} dt = O \left( \frac{1}{(u-v)^2} \right). \quad (12)$$

• In the case  $u = v$ , we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2} dt \\ &= \int_0^{+\infty} \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} \frac{dt}{\sqrt{u}} \\ &= \frac{\log \frac{1}{u}}{\sqrt{u}} \int_0^{+\infty} e^{-t^2} dt + \frac{1}{\sqrt{u}} \int_0^{+\infty} \log \left( u \left( \frac{\lambda}{2} + \mu \right)^2 + \lambda^2 t^2 \right) e^{-t^2} dt \\ &= \frac{\sqrt{\pi}}{2} \frac{\log \frac{1}{u}}{\sqrt{u}} + \frac{1}{\sqrt{u}} \{N_1 + N_2\}, \end{aligned} \quad (13)$$

with

$$N_1 = \int_0^{+\infty} \log(\lambda^2 t^2) e^{-t^2} dt$$

and

$$N_2 = \int_0^{+\infty} \log \left( 1 + \frac{u \left( \frac{1}{2} + \frac{\mu}{\lambda} \right)^2}{t} \right) e^{-t^2} dt.$$

We have

$$\begin{aligned}
|N_1| &= \int_0^{+\infty} \log(\lambda^2) e^{-t^2} dt + \int_0^{+\infty} \log(t^2) e^{-t^2} dt \\
&= 2 \log \lambda \int_0^{+\infty} e^{-t^2} dt + 2 \int_0^{+\infty} \log t e^{-t^2} \frac{dt}{2\sqrt{t}} \\
&= 2\sqrt{\pi} \log \lambda + \frac{1}{2} \Gamma'(\frac{1}{2}) \\
&= \sqrt{\pi} \log \left( \frac{\lambda^2}{4} \right) - \frac{\sqrt{\pi}}{2} \gamma_0.
\end{aligned} \tag{14}$$

Furthermore,

$$\begin{aligned}
|N_2| &= \int_0^{\sqrt{u}} \log \left( 1 + \frac{u(\frac{1}{2} + \frac{\mu}{\lambda})^2}{t^2} \right) e^{-t^2} dt + \int_{\sqrt{u}}^{+\infty} \log \left( 1 + \frac{u(\frac{1}{2} + \frac{\mu}{\lambda})^2}{t^2} \right) dt \\
&\leq \int_0^{\sqrt{u}} \log \left( 1 + \frac{u(\frac{1}{2} + \frac{\mu}{\lambda})^2}{t^2} \right) e^{-t^2} dt + u(\frac{1}{2} + \frac{\mu}{\lambda})^2 \int_{\sqrt{u}}^{+\infty} \frac{e^{-t^2}}{t^2} dt \\
&= \sqrt{u} \log \left( 1 + (\frac{1}{2} + \frac{\mu}{\lambda})^2 \right) + \int_0^{\sqrt{u}} \frac{2u(\frac{1}{2} + \frac{\mu}{\lambda})^2}{t^2 + u(\frac{1}{2} + \frac{\mu}{\lambda})^2} dt \\
&\quad + u(\frac{1}{2} + \frac{\mu}{\lambda})^2 \int_{\sqrt{u}}^{+\infty} \frac{e^{-t^2}}{t^2} dt \\
&= \sqrt{u} \log \left( 1 + (\frac{1}{2} + \frac{\mu}{\lambda})^2 \right) + 2\sqrt{u} (\frac{1}{2} + \frac{\mu}{\lambda})^2 + u(\frac{1}{2} + \frac{\mu}{\lambda})^2 \int_{\sqrt{u}}^{+\infty} \frac{e^{-t^2}}{t^2} dt \\
&\ll \sqrt{u}.
\end{aligned} \tag{15}$$

Using equations (13), (14) and (15), we get

$$\int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2} dt = \sqrt{\frac{\pi}{4u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \sqrt{\frac{\pi}{u}} \log \left( \frac{\lambda^2}{4} \right) + O(1). \tag{16}$$

• In the case  $v = 0$ , we have

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2+itu} dt \\
&= \int_0^{+\infty} \log \left( \left( \frac{\lambda}{2} + \mu \right)^2 + \frac{\lambda^2 t^2}{u} \right) e^{-t^2} \cos(\sqrt{ut}) \frac{dt}{\sqrt{u}} \\
&= \frac{\log \frac{1}{\sqrt{u}}}{\sqrt{u}} \int_0^{+\infty} e^{-t^2} \cos(\sqrt{ut}) dt + \frac{1}{\sqrt{u}} \int_0^{+\infty} \log(\lambda^2 t^2) e^{-t^2} \cos(\sqrt{ut}) dt \\
&\quad + \frac{1}{\sqrt{u}} \int_0^{+\infty} \log \left( 1 + \frac{u}{t^2} \left( \frac{1}{2} + \frac{\mu}{\lambda} \right) \right) e^{-t^2} \cos(\sqrt{ut}) dt \\
&= \frac{\log \frac{1}{\sqrt{u}}}{\sqrt{u}} P_1 + \frac{1}{\sqrt{u}} \{P_2 + P_3\}.
\end{aligned} \tag{17}$$

With the equality

$$\cos(\sqrt{ut}) = 1 + O(ut^2),$$

we obtain

$$P_1 = \int_0^{+\infty} e^{-t^2} dt + O \left( \int_0^{+\infty} t^2 e^{-t^2} dt \right) = \frac{\sqrt{\pi}}{2} + O(u) \tag{18}$$

and

$$\begin{aligned}
P_2 &= \int_0^{+\infty} \log(\lambda^2 t^2) e^{-t^2} dt + O \left( u \int_0^{+\infty} \log(\lambda^2 t^2) t^2 e^{-t^2} dt \right) \\
&= \sqrt{\pi} \log \left( \frac{\lambda^2}{4} \right) - \frac{\sqrt{\pi}}{2} \gamma_0 + O(u).
\end{aligned} \tag{19}$$

Furthermore, we have

$$|P_3| \leq N_2 = O(\sqrt{u}). \tag{20}$$

Hence, using (15), (16), (17) and (18), we obtain

$$\int_{-\infty}^{+\infty} \log \left| \left( \frac{\lambda}{2} + \mu \right) + i\lambda t \right| e^{-ut^2+itu} dt = \sqrt{\frac{\pi}{4u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \sqrt{\frac{\pi}{u}} \log \left( \frac{\lambda^2}{2} \right) + O(1), \tag{21}$$

and Lemma 2.2 follows.  $\square$

*Proof. (of Theorem 1.1).* First, it is easy to verify that

$$0 \leq \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) \leq 1,$$

hence

$$0 \leq I(\lambda_j, \mu_j) \leq 2 \int_0^{+\infty} \frac{e^{-(v+\lambda x)^2/4u}}{\sqrt{4\pi u}} dx \ll 1.$$

Second, we consider the asymptotic behavior of the quantity

$$\frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \Lambda_F(n) e^{-\frac{(v+\log n)^2}{4u}} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{n} e^{-\frac{(v-2u-\log n)^2}{4u}}, \quad (22)$$

in Lemma 2.1. The behavior depends on the choice of  $v$ . For the case  $v = 0$  and  $0 < u < 1$ , (22) is of exponential decay as  $u \rightarrow 0^+$ . For the case  $v = -\log m$ ,  $m \in \mathbb{N} \setminus \{0, 1\}$ , and  $0 < u < 1$ , we have

$$\begin{aligned} & \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \Lambda_F(n) e^{-\frac{(v+\log n)^2}{4u}} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{n} e^{-\frac{(v-2u-\log n)^2}{4u}} \\ &= -\frac{\Lambda_F(m)}{\sqrt{4\pi u}} + O\left(\frac{e^{-(\log 2)^2/8u}}{\sqrt{u}} \sum_{n \geq 2} \frac{\Lambda_F(n)}{n} e^{-(\log n)^2/8}\right) \\ & \quad + e^{-\frac{1}{8u}(-\log m + \log(m+1))} \left( \sum_{m \neq n=2}^{m^2} \Lambda_F(n) + \sum_{n > m^2} \Lambda_F(n) e^{-(\log n)^2/32} \right). \end{aligned}$$

Similarly asymptotic behaviour for (22) can be obtained for other  $v$ . Combining this with Lemmas 2.1 and 2.2, we obtain the assertion of Theorem 1.1.  $\square$

The asymptotic formula in *i*) is another version of the asymptotic formula of  $N_F(T)$ , number of non-trivial zeros  $\rho$  with  $0 < \gamma < T$ . To see this, we consider the case  $v = u$  in *(i)* under the Generalized Riemann Hypothesis, then the asymptotic formula in *(i)* is

$$\sum_{\gamma} e^{-u(1/4+\gamma^2)} = \frac{d_F}{\sqrt{16\pi u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \frac{1}{\sqrt{4\pi u}} \log \left( \frac{q_F}{(4\pi)^{d_F}} \right) + O(1).$$

By integration by parts, it follows that

$$-\int_0^{+\infty} N_F(T) d(e^{-uT^2}) = \frac{d_F}{2\sqrt{16\pi u}} \left( \log \frac{1}{u} - \gamma_0 \right) + \frac{1}{2\sqrt{4\pi u}} \log \left( \frac{q_F}{(4\pi)^{d_F}} \right) + O(1).$$

The asymptotic formula in *ii*) may be regarded as a smooth version of (1) with the measure given by the Gaussian function.



### 3 Proof of Theorem 1.4

The proof is an analogous of the argument used by Kamya in [7, Theorem 1].

**Case:**  $0 < u < 1$  and  $v \in \mathbb{R}^*$ .

Let us denote the right-hand side of (3) by  $T_1 + T_2 + m_F T_3 + m_F T_4 + T_5 + T_6 + T_7 + T_8$ .

We know that there exists a constant  $c > 0$  such that

$$|f(x)| \leq c e^{-(\frac{1}{2}+b)|x|}. \quad (23)$$

This gives

$$T_5, T_6 = O(u), 0 < u < 1. \quad (24)$$

From (23), we deduce that

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} f\left(\frac{x-v}{u}\right) e^{\frac{x}{2}} dx \right| &\leq c \int_{-\infty}^{+\infty} e^{-(\frac{1}{2}+b)|\frac{x-v}{u}|} e^{\frac{x}{2}} dx \\ &= c u e^{v/2} \int_{-\infty}^{+\infty} e^{-(\frac{1}{2}+b)|x|} e^{ux/2} dx \\ &\leq c u e^{v/2} \int_{-\infty}^{+\infty} e^{-b|x|} dx \\ &= O(u). \end{aligned}$$

Hence

$$T_3, T_4 = O(u), 0 < u < 1. \quad (25)$$

By using conditions  $c'$ , (3) and  $0 < x \leq |v|/2$ , we can easily prove that

$$\left| f\left(\frac{x-v}{u}\right) - f\left(\left(\frac{-v}{u}\right)^+\right) \right| \leq (2cD)^{1/2} \frac{e^{-(\frac{1}{2}+b)\frac{|v|}{4u}}}{u^{\epsilon/2}} x^{\epsilon/2}. \quad (26)$$

Therefore

$$\begin{aligned}
|T_7| &\leq \int_0^{|v|/2} \left( f\left(\frac{-\lambda_j x - v}{u}\right) - f\left(\left(\frac{-v}{u}\right)^+\right) \right) \frac{e^{-(\frac{\lambda}{2}+\mu)x}}{1-e^{-x}} dx \\
&\quad + \int_{|v|/2}^{+\infty} \left( f\left(\frac{-\lambda_j x - v}{u}\right) - f\left(\left(\frac{-v}{u}\right)^+\right) \right) \frac{e^{-(\frac{\lambda}{2}+\mu)x}}{1-e^{-x}} dx \\
&\leq (2cD)^{1/2} \frac{e^{-(\frac{1}{2}+b)\frac{|v|}{4u}}}{u^{\epsilon/2}} \int_0^{|v|/2} x^{\epsilon/2} \frac{e^{-(\frac{\lambda}{2}+\mu)x}}{1-e^{-x}} dx \\
&\quad + c \int_{|v|/2}^{+\infty} \left( e^{-(\frac{1}{2}+b)|x-v|/u} + e^{-(\frac{1}{2}+b)|v|/u} \right) \frac{e^{-(\frac{\lambda}{2}+\mu)x}}{1-e^{-x}} dx \\
&\leq \frac{c}{1-e^{|v|/2}} \int_{|v|/2}^{+\infty} e^{-(\frac{\lambda}{2}+\mu)x} e^{-(\frac{1}{2}+b)|x-v|/u} dx + O(u). \tag{27}
\end{aligned}$$

For  $0 < u < 1$ , we have

$$\begin{aligned}
\int_{|v|/2}^{+\infty} e^{-(\frac{\lambda}{2}+\mu)x} e^{-(\frac{1}{2}+b)|x-v|/u} dx &\leq e^{-v/2} \int_{-\infty}^{+\infty} e^{-(\frac{\lambda}{2}+\mu)x} e^{-(\frac{1}{2}+b)|x|/u} dx \\
&= u e^{-v/2} \int_{-\infty}^{+\infty} e^{-(\frac{\lambda}{2}+\mu)x} e^{-(\frac{1}{2}+b)|x|} dx \\
&\leq u e^{-v/2} \int_{-\infty}^{+\infty} e^{-b|x|} dx \\
&= O(u).
\end{aligned}$$

Consequently, we obtain

$$T_7 = O(u), \quad 0 < u < 1. \tag{28}$$

The same argument used above yields

$$T_8 = O(u), \quad 0 < u < 1. \tag{29}$$

- If  $v \neq \pm \log m$ ,  $m = 1, 2, \dots$  and  $0 < u < 1/2$ , by (3), we have

$$\begin{aligned}
|T_1 + T_2| &\leq c \sum_{n=2}^{+\infty} \frac{|\Lambda_F(n)|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\frac{|\log n - v|}{u}} + c \sum_{n=2}^{+\infty} \frac{|\overline{\Lambda_F(n)}|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\frac{|\log n + v|}{u}} \\
&\leq c e^{-(\frac{1}{2}+b)\delta/2u} \sum_{n=2}^{+\infty} \frac{|\Lambda_F(n)|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\frac{|\log n - v|}{u}} + c e^{-(\frac{1}{2}+b)\delta/2u} \sum_{n=2}^{+\infty} \frac{|\overline{\Lambda_F(n)}|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\frac{|\log n + v|}{u}} \\
&\leq c e^{-(\frac{1}{2}+b)\delta/2u} \left\{ \sum_{n=2}^{+\infty} \frac{|\Lambda_F(n)|}{\sqrt{n}} e^{-(\frac{1}{2}+b)|\log n - v|} + \sum_{n=2}^{+\infty} \frac{|\overline{\Lambda_F(n)}|}{\sqrt{n}} e^{-(\frac{1}{2}+b)|\log n + v|} \right\},
\end{aligned}$$

where  $\delta$  is the distance between the set  $\{\pm \log m, m = 1, 2, \dots\}$  and  $\{v\}$ . We may assume that  $v > 0$  to estimate the right-hand side of the last expression. This gives

$$\begin{aligned}
|T_1 + T_2| &\leq c e^{-(\frac{1}{2}+b)\delta/2u} \left\{ \sum_{2 \leq n \leq e^v} \frac{|\Lambda_F(n)|}{\sqrt{n}} + e^{(\frac{1}{2}+b)v} \sum_{e^v < n} \frac{|\Lambda_F(n)|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\log n} \right\} \\
&\quad + c e^{-(\frac{1}{2}+b)\delta/2u} \sum_{n=2}^{+\infty} \frac{|\Lambda_F(n)|}{\sqrt{n}} e^{-(\frac{1}{2}+b)\log n} \\
&= O \left( e^{-(\frac{1}{2}+b)\delta/2u} \left( 1 + \sum_{n=2}^{+\infty} \frac{\Lambda_F(n)}{n^{1+b}} \right) \right) \\
&= O(u).
\end{aligned}$$

If  $v = \log m$ ,  $m = 2, 3, \dots$ , we pick up the term  $-\frac{\Lambda_F(m)}{\sqrt{m}}f(0)$  from  $T_1 + T_2$ , the remainder sums are dominated by  $O(u)$  with the same argument as above. While, for  $v = -\log m$ ,  $m = 2, 3, \dots$  we pick up again the term  $-\frac{\Lambda_F(m)}{\sqrt{m}}f(0)$  from  $T_1 + T_2$ . Hence, for  $u \mapsto 0^+$ ,

$$T_1 + T_2 = \begin{cases} O(u) & \text{if } v \neq \pm \log m, m = 1, 2, \dots \\ -\frac{\Lambda_F(m)}{\sqrt{m}}f(0) + O(u) & \text{if } v = \log m, m = 2, 3, \dots \\ -\frac{\Lambda_F(m)}{\sqrt{m}}f(0) + O(u) & \text{if } v = -\log m, m = 2, 3, \dots \end{cases} \quad (30)$$

**Case:**  $v = 0$ .

Formula (2) is valid for all  $0 < u < 1$  and  $v = 0$  under conditions a), b) and c). Let

$$f(x) = \frac{f(x^+) + f(x^-)}{2}.$$

Then, formula (3) can be rewritten in the form

$$\begin{aligned}
& \sum_{\rho} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{x(\rho-\frac{1}{2})} dx \\
&= - \sum_{n=2}^{+\infty} \frac{\Lambda_F(n)}{\sqrt{n}} f\left(\frac{\log n}{u}\right) - \sum_{n=2}^{+\infty} \frac{\overline{\Lambda_F(n)}}{\sqrt{n}} f\left(\frac{-\log n}{u}\right) \\
&\quad + m_F \left( \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{\frac{x}{2}} dx + \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{-\frac{x}{2}} dx \right) + 2f(0) \log Q \\
&\quad + \sum_{j=1}^r \lambda_j \left( \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \overline{\mu_j} \right) \right) f(0) \\
&\quad - \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( f\left(\frac{-\lambda_j x}{u}\right) - f(0^-) \right) \frac{e^{-(\frac{\lambda_j}{2} + \mu_j)x}}{1 - e^{-x}} dx \\
&\quad - \sum_{j=1}^r \lambda_j \int_0^{+\infty} \left( f\left(\frac{\lambda_j x}{u}\right) - f(0^+) \right) \frac{e^{-(\frac{\lambda_j}{2} + \overline{\mu_j})x}}{1 - e^{-x}} dx
\end{aligned} \tag{31}$$

Denote the right-hand side of (31) by  $M_1 + M_2 + m_F M_3 + m_F M_4 + M_5 + M_6 + M_7$ . By (23) and the same arguments of the first case, we have

$$M_1, M_2 = O(u), \quad 0 < u < 1/2, \tag{32}$$

and

$$M_3, M_4 = O(u), \quad 0 < u < 1. \tag{33}$$

Now, let treat the term  $M_6$ . We have

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) \frac{e^{-(\frac{\lambda}{2} + \mu)x}}{1 - e^{-x}} dx \\
&= - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) e^{-(\frac{\lambda}{2} + \mu)x} \frac{dx}{x} \\
&\quad + f(0) \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-(\frac{\lambda}{2} + \mu)x} dx \\
&\quad - \int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-(\frac{\lambda}{2} + \mu)x} f\left(\frac{-\lambda x}{u}\right) dx.
\end{aligned} \tag{34}$$

Let

$$H(x) = \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-(\frac{\lambda}{2} + \mu)x}.$$

Then, the quantity given by (34) is equal to

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) e^{-(\frac{\lambda}{2} + \mu)x} \frac{dx}{x} + f(0) \int_0^{+\infty} H(x) dx \\
& - \int_0^{+\infty} H(x) f\left(\frac{-\lambda x}{u}\right) dx
\end{aligned} \tag{35}$$

Because  $H(x)$  is bounded on  $(0, \infty)$ , from (23) and integration by part,

$$\int_0^{+\infty} H(x) f\left(\frac{-\lambda x}{u}\right) dx = O(u).$$

By observing that

$$\int_0^{+\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} + 1 \right) e^{-(\frac{\lambda}{2} + \mu)x} dx = \log\left(\frac{\lambda}{2} + \mu\right) - \frac{\Gamma'}{\Gamma}\left(\frac{\lambda}{2} + \mu\right),$$

the second term of (35) is

$$f(0) \log\left(\frac{\lambda}{2} + \mu\right) - f(0) \frac{\Gamma'}{\Gamma}\left(\frac{\lambda}{2} + \mu\right). \tag{36}$$

Hence

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) \frac{e^{-(\frac{\lambda}{2} + \mu)x}}{1 - e^{-x}} dx \\
& = - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) e^{-(\frac{\lambda}{2} + \mu)x} \frac{dx}{x} \\
& \quad + f(0) \log\left(\frac{\lambda}{2} + \mu\right) - f(0) \frac{\Gamma'}{\Gamma}\left(\frac{\lambda}{2} + \mu\right).
\end{aligned} \tag{37}$$

Using the fact (see. [5]),

$$\int_0^{+\infty} (e^{-x} - e^{-\omega x}) \frac{dx}{x} = \log \omega,$$

we get

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) e^{-(\frac{\lambda}{2} + \mu)x} \frac{dx}{x} + f(0) \log\left(\frac{\lambda}{2} + \mu\right) \\
& = - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) e^{-(\frac{\lambda}{2} + \mu)x} \frac{dx}{x} \\
& \quad + f(0) \int_0^{+\infty} \left( e^{-x} - e^{-(\frac{\lambda}{2} + \mu)x} \right) \frac{dx}{x} \\
& = - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) e^{-(\frac{\lambda}{2} + \mu)x} - f(0^-) e^{-x} \right) \frac{dx}{x} \\
& = - \int_0^{+\infty} \left( f(-\lambda x) e^{-(\frac{\lambda}{2} + \mu)ux} - f(0^-) e^{-ux} \right) \frac{dx}{x} \\
& = f(0) \int_0^{+\infty} \left( e^{-(\frac{\lambda}{2} + \mu)ux} - e^{-ux} \right) \frac{dx}{x} - \int_0^{+\infty} (f(-\lambda x) - f(0^-) e^{-x}) \frac{dx}{x} \\
& \quad + \int_0^{+\infty} \left( 1 - e^{-(\frac{\lambda}{2} + \mu)ux} \right) f(-\lambda x) \frac{dx}{x}. \tag{38}
\end{aligned}$$

The first term of (38) is

$$-f(0) \log \left[ \left( \frac{\lambda}{2} + \mu \right) u \right],$$

while the third term is  $O(u)$ . Therefore

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{-\lambda x}{u}\right) - f(0^-) \right) \frac{e^{-(\frac{\lambda}{2} + \mu)x}}{1 - e^{-x}} dx \\
& = -f(0) \left[ \log \left( \left( \frac{\lambda}{2} + \mu \right) u \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda}{2} + \mu \right) \right] \\
& \quad - \int_0^{+\infty} (f(-\lambda x) - f(0^-) e^{-x}) \frac{dx}{x}. \tag{39}
\end{aligned}$$

The same argument used to prove (39) gives

$$\begin{aligned}
& - \int_0^{+\infty} \left( f\left(\frac{\lambda x}{u}\right) - f(0^+) \right) \frac{e^{-(\frac{\lambda}{2} + \bar{\mu})x}}{1 - e^{-x}} dx \\
& = -f(0) \left[ \log \left( \left( \frac{\lambda}{2} + \bar{\mu} \right) u \right) + \frac{\Gamma'}{\Gamma} \left( \frac{\lambda}{2} + \bar{\mu} \right) \right] \\
& \quad - \int_0^{+\infty} (f(\lambda x) - f(0^+) e^{-x}) \frac{dx}{x}. \tag{40}
\end{aligned}$$

This ends the proof of Theorem 1.4.

## 4 Proof of Theorem 1.5

Assume that the Riemann hypothesis holds, that is, all non-trivial zeros have the form  $\rho = 1/2 + i\gamma$ . Let  $f$  be a function as above, then

$$\widehat{f}(t) = O\left(\frac{1}{|t|}\right).$$

Let  $u$  be such that  $0 < u < 1$  and denote the  $n^{\text{th}}$  value of  $\gamma > 0$  by  $\gamma_n$ . Bombieri and Hejhal in [1] proved that

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_1 T + O(\log T), \quad (41)$$

where

$$c_1 = \frac{1}{2\pi} (\log q_F - d_F(\log 2\pi + 1)) \text{ and } q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}.$$

Using the same argument as in Guinand [4, page 108], we can prove that

$$\begin{aligned} & - \int_0^{\gamma_{N+1}} \left( N_F(T) - \frac{d_F}{2\pi} T \log \left( \frac{T}{2\pi} \right) - \frac{1}{2\pi} T (\log q_F - d_F) \right) \frac{d}{dT} \left( \widehat{f}(-uT) \right) dT \\ & = \sum_{r=1}^N \widehat{f}(-u\gamma_r) - \frac{d_F}{2\pi} \int_0^{\gamma_{N+1}} \widehat{f}(-uT) \log \left( \frac{T}{2\pi} \right) dT + O\left( \frac{\log \gamma_{N+1}}{\gamma_{N+1}} \right). \end{aligned}$$

Consequently,

$$\sum_{r=1}^{+\infty} \widehat{f}(-u\gamma_r) = \frac{1}{2u} \sum_{\gamma} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{i\gamma x} dx,$$

which is finite. Therefore

$$\begin{aligned} & - \int_0^{\infty} \left( N_F(T) - \frac{d_F}{2\pi} T \log \left( \frac{T}{2\pi} \right) - \frac{1}{2\pi} T (\log q_F - d_F) \right) \frac{d}{dT} \left( \widehat{f}(-uT) \right) dT \\ & = \frac{1}{2u} \sum_{\gamma} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{i\gamma x} dx - \frac{d_F}{2\pi} \int_0^{+\infty} \widehat{f}(-uT) \log \left( \frac{T}{2\pi} \right) dT \\ & \quad + O\left( \frac{\log \gamma_{N+1}}{\gamma_{N+1}} \right). \end{aligned}$$

With the change of variable  $T = \frac{t}{\lambda}$  and by the Fourier inversion formula, we obtain

$$\begin{aligned}
& -\frac{d_F}{2\pi} \int_0^{+\infty} \widehat{f}(-uT) \log\left(\frac{T}{2\pi}\right) dT \\
&= -\frac{d_F}{2\pi\lambda} \int_0^{+\infty} \widehat{f}\left(-u\frac{t}{\lambda}\right) \log\left(\frac{t}{2\pi\lambda}\right) dt \\
&= -\frac{d_F}{2\pi\lambda} \int_0^{+\infty} \widehat{f}\left(-u\frac{t}{\lambda}\right) (\log t - \log(2\pi\lambda)) dt \\
&= \frac{d_F \log(2\pi u)}{2u} f(0) - \frac{d_F}{2\pi u \lambda} \int_0^{+\infty} \widehat{f}(-t) \log t dt.
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{\gamma} \int_{-\infty}^{+\infty} f\left(\frac{x}{u}\right) e^{i\gamma x} dx \\
&= -d_F \log(2\pi u) f(0) + \frac{d_F}{\pi\lambda} \int_0^{+\infty} \widehat{f}\left(-\frac{T}{\lambda}\right) \log T dT \\
& -2u \int_0^{\infty} \left( N_F(T) - \frac{d_F}{2\pi} T \log\left(\frac{T}{2\pi}\right) - \frac{1}{2\pi} T (\log q_F - d_F) \right) \frac{d}{dT} \left( \widehat{f}(-uT) \right) dT.
\end{aligned} \tag{42}$$

The second term in the right-hand-side of the last equation (42) is given by Lemma 1.6. Then, Theorem 1.5 follows.

## 5 Appendix: the Weil explicit formulas and the Li coefficients

In 1997, Xian-Jin Li has discovered a new positivity criterion for the Riemann hypothesis. In [9] he proved that the Riemann hypothesis is equivalent with the non-negativity of numbers

$$\lambda_n = \sum_{\rho} \left( 1 - \left( 1 - \frac{1}{\rho} \right)^n \right)$$

for all  $n \in \mathbb{N}$ , where the sum is taken over all non-trivial zeros of the Riemann zeta function. A little later, Bombieri and Lagarias [2] observed that the Li criterion can be generalized to a multi-set of complex numbers satisfying



certain conditions, and gave an arithmetic formula for numbers  $\lambda_n$ . In [14] and [15], it was shown that one could formulate a Li-type criterion for a general class of Dirichlet series, which includes elements of the Selberg class and obtain an arithmetic formula for the generalized Li coefficient defined below.

In this appendix, we give another form of the Weil explicit formulas and use it to find an arithmetic formula for the generalized Li coefficients.

By  $\varphi BV$  we denote the set of functions of bounded  $\varphi$ -variation in the sense of L. C. Young. A function  $f$  is said to be of  $\varphi$  bounded variation on an interval  $I$  with the end points  $a$  and  $b$  if

$$V_\varphi(f, I) = \sup \sum_n \varphi(|f(I_n)|) < \infty,$$

where  $f(I)$  stands for  $f(b) - f(a)$  and the supremum is taken over all systems  $\{I_n\}$  of non overlapping subintervals of  $I$ .

**Proposition 5.1.** *Let a regularized function  $G$  fulfill the following conditions:*

1.  $G \in \varphi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ .
2.  $G(x)e^{1/2+\epsilon}|x| \in \varphi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ , for some  $\epsilon > 0$ .
3.  $G(x) + G(-x) - 2G(0) = O(|\log|x||^\alpha)$ , as  $x \rightarrow 0$ , for some  $\alpha > 2$ .

Let  $F(s) \in \mathcal{S}$ . Then,

$$\begin{aligned} \sum_{\rho} \tilde{g}_{1/2}(\rho) &= m_F(\tilde{g}_{1/2}(0) + \tilde{g}_{1/2}(1)) \\ &\quad - \sum_n \frac{b_F(n)}{n^{1/2}} g(n) - \sum_n \frac{\overline{b_F}(n)}{n^{1/2}} g(1/n) + 2G(0) \log Q_F \\ &\quad + \sum_{j=1}^r \int_0^{+\infty} \left\{ \frac{2\lambda_j G_j(0)}{x} - \frac{e^{\left(\left(1-\frac{\lambda_j}{2}-\Re(\mu_j)\right)\frac{x}{\lambda_j}\right)}}{1 - e^{-\frac{x}{\lambda_j}}} (G_j(x) + G_j(-x)) \right\} e^{-\frac{x}{\lambda_j}} dx, \end{aligned}$$

where  $\rho$  runs over all non-trivial zeros of  $F(s)$  counted with multiplicity and  $\tilde{g}_{1/2}$  denotes the translate by  $1/2$  of the Mellin transform of the function  $g$ .

Let  $F$  be a function in the Selberg class non-vanishing at  $s = 1$  and let us define the xi-function  $\xi_F(s)$  by

$$\xi_F(s) = s^{m_F}(s-1)^{m_F}\phi_F(s).$$

The function  $\xi_F(s)$  satisfies the functional equation

$$\xi_F(s) = \omega \overline{\xi_F(1-\bar{s})}.$$

The function  $\xi_F$  is an entire function of order 1. Therefore, by the Hadamard product, it can be written as

$$\xi_F(s) = \xi_F(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

where the product is over all zeros of  $\xi_F(s)$  in the order given by  $|\Im(\rho)| < T$  for  $T \rightarrow \infty$ . Let  $\lambda_F(n)$ ,  $n \in \mathbb{Z}$ , be a sequence of numbers defined by a sum over the non-trivial zeros of  $F(s)$  as

$$\lambda_F(n) = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right],$$

where the sum over  $\rho$  is

$$\sum_{\rho} = \lim_{T \rightarrow \infty} \sum_{|\Im \rho| \leq T}.$$

These coefficients are expressible in terms of power-series coefficients of functions constructed from the  $\xi_F$ -function. For  $n \leq -1$ , the Li coefficients  $\lambda_F(n)$  correspond to the following Taylor expansion at the point  $s = 1$

$$\frac{d}{dz} \log \xi_F \left( \frac{1}{1-z} \right) = \sum_{n=0}^{+\infty} \lambda_F(-n-1) z^n$$

and for  $n \geq 1$ , they correspond to the Taylor expansion at  $s = 0$

$$\frac{d}{dz} \log \xi_F \left( \frac{-z}{1-z} \right) = \sum_{n=0}^{+\infty} \lambda_F(n+1) z^n.$$

Let  $\mathcal{Z}$  the multi-set of zeros of  $\xi_F(s)$  (counted with multiplicity). The multi-set  $\mathcal{Z}$  is invariant under the map  $\rho \mapsto 1 - \bar{\rho}$ . We have

$$1 - \left(1 - \frac{1}{\rho}\right)^{-n} = 1 - \left(\frac{\rho-1}{\rho}\right)^{-n} = 1 - \left(\frac{-\rho}{1-\rho}\right)^n = 1 - \overline{\left(1 - \frac{1}{1-\bar{\rho}}\right)^n}$$

and this gives the symmetry  $\lambda_F(-n) = \overline{\lambda_F(n)}$ . Using the corollary in [2, Theorem 1], we get the following generalization of the Li criterion for the Riemann hypothesis.

**Theorem 5.2.** [14] [16] *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  non-vanishing at  $s = 1$ . Then, all non-trivial zeros of  $F(s)$  lie in the line  $\Re(s) = 1/2$  if and only if  $\Re(\lambda_F(n)) > 0$  for  $n = 1, 2, \dots$*

Let consider the following hypothesis:

**$\mathcal{H}$ :** **there exists a constant  $c > 0$  such that  $F(s)$  is non-vanishing in the region:**

$$\left\{ s = \sigma + it; \sigma \geq 1 - \frac{c}{\log(Q_F + 1 + |t|)} \right\}.$$

Let Consider  $g(x) = G(-\log x)$ , for  $x > 0$  and  $G_j(x) = G(x)e^{\left(\frac{ix\Im(\mu_j)}{\lambda_j}\right)}$ . Applying Proposition 5.1 to the function

$$G_{n,z}(x) = \begin{cases} e^{-(z+x/2)} \sum_{l=1}^n \binom{n}{l} \frac{(-x)^{l-1}}{(l-1)!} & \text{if } x > 0, \\ n/2 & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where  $z$  is a positive constant. We obtain the following Theorem.

**Theorem 5.3.** [14] [15] *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  satisfying  $\mathcal{H}$ . Then, we have*

$$\begin{aligned} \lambda_F(-n) &= m_F + n(\log Q_F - \frac{d_F}{2}\gamma_0) \\ &- \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k} (\log k)^{l-1} - \frac{m_F}{l} (\log X)^l \right\} \\ &+ n \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \\ &- \sum_{j=1}^r \sum_{k=2}^n \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \left( \frac{1}{l + \lambda_j + \mu_j} \right)^k, \end{aligned} \tag{43}$$

where  $\gamma_0$  is the Euler constant.

In a recent works [10] and [17], new asymptotic formula was given for the generalized Li coefficients. To do so, we used two different methods, both of them give the same main term. The first is inspired from Lagarias method yields to a sharper error term  $O(\sqrt{n} \log n)$ , while the second use the saddle-point method.

**Theorem 5.4.** [10] [17] *Let  $F \in \mathcal{S}$ . Then*

$$RH \Leftrightarrow \lambda_F(-n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

where

$$c_F = \frac{d_F}{2}(\gamma_0 - 1) + \frac{1}{2} \log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$$

and  $\gamma_0$  is the Euler constant.

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